The preceding considerations lead to some direct extensions of the most fundamental rules of operating with positive and negative numbers. For example, the rule for the sums of two real numbers with opposite signs is included in the following; To obtain the sum of a number on a given ray and a number on the extended ray* find the difference of their absolute values and prefix to this the angle of that one whose absolute value is the greater. The fact that the sign of the product of a positive and a negative number is negative is included in the statement that the angle of the product of a number on the $a_0$ ray and a number on its extension is $2a_0+\pi$. From this it results that the necessary and sufficient condition that a number is real is that a negative number is obtained by multiplying it by a number on the extension of its ray, and the necessary and sufficient condition that a number is a pure imaginary is that a positive number is obtained when we multiply the number into any number on the extended ray.

In the light of these results the question whether the terms positive and negative, which do not exhibit any evidence of the fact that they represent two special values of the possible amplitudes of a number, should be replaced, at least in theoretic work, by terms which exhibit their places in the infinite series of amplitudes, assumes a deeper meaning. If one should be asked to defend the terms 0-numbers and $\pi$-numbers it would be merely necessary to reply that the adjectives express the values of the amplitudes of these numbers, but no such rational defense could be made for the terms positive numbers and negative numbers. At any rate this view point seems to deserve notice since the question involved is so fundamental, and the present note makes no other claims for usefulness or novelty than the presentation of a very elementary and important matter in a somewhat new light.

---

## ON CERTAIN PROPERTIES OF THE ORBITS OF A PARTICLE SUBJECT TO A CENTRAL FORCE VARYING AS AN INTEGRAL POWER OF THE DISTANCE.†

By E. J. MOULTON and F. H. HODGE, The University of Chicago.

In this paper are discussed the orbits of a particle moving subject to a central force varying as an integral power of the distance. The discussion is made with regard to their concavity and convexity, the number and distances of their apses, the ranges of values through which their radial dis-

*If the angles of two rays differ by $\pi$ each one of them is said to be the extension of the other.

†This discussion had its origin in an exercise given by Professor F. R. Moulton to his class in Celestial Mechanics. It was read before the Chicago Section of the American Mathematical Society, April 18, 1908.

tances may vary, the amplitudes of the angles described by the radii vectores, and the time required for the maximum variation of the radial distance of the particle from the center of force. The discussion will be made from the implicit relations among the variables obtained from the first integrals.

It is shown in works on mechanics that the differential equations of the motion of a particle subject to a central force are

$$(1)\qquad \begin{cases} f=h^2u^2\left(n+\dfrac{d^2u}{d\theta^2}\right), & (n\geq 0), \\ \dfrac{d\theta}{dt}=hu^2, \end{cases}$$

where $f$ is the force, $h$ is the constant of areas, $u$ is the reciprocal of the radial distance to the particle, $t$ is the time, and $\theta$ is the angle between the radius vector and some arbitrarily chosen reference line, the origin being at the center of force. If the force varies as an integral power of the distance we have

$$(2)\qquad f=\pm m^2h^2u^a,$$

where $m^2h^2$ is an arbitrary constant not equal to zero, and $a$ is an integer. The positive sign signifies that it is an attractive force; the negative sign a repulsive force. From (1) and (2) we obtain

$$(3)\qquad \begin{cases} u+\dfrac{d^2u}{d\theta^2}=\pm m^2u^{a-2} & (u\geq 0), \\ \dfrac{d\theta}{dt}=hu^2, \end{cases}$$

as the differential equations of motion.

Multiplying both sides of the first equation of (3) by $2\dfrac{du}{d\theta}$, and integrating, it is found that

$$(4)\qquad \begin{cases} u^2+\left(\dfrac{du}{d\theta}\right)^2=\pm\dfrac{2m^2}{a-1}u^{a-1}+c, & (a\neq 1,\ u\geq 0), \\ u^2+\left(\dfrac{du}{d\theta}\right)^2=\pm 2m^2\log u+c, & (a=1,\ u\geq 0), \end{cases}$$

where $c$ is a constant of integration which is determined by the initial conditions. We then have

$$(5)\qquad \begin{cases} \theta = \int_{u_0}^{u} \dfrac{du}{\sqrt{\pm\dfrac{2m^2}{\alpha-1}u^{\alpha-1}-u^2+c}}, & (\alpha\neq 1), \\ \theta = \int_{u_0}^{u} \dfrac{du}{\sqrt{\pm 2m^2\log u-u^2+c}}, & (\alpha=1). \end{cases}$$

It is seen that when $\alpha=-1$, $+2$, 3, 4, or 5, $u$ is a trigonometric, exponential, or elliptic function of $\theta$, and the characteristics of the orbits are readily obtained. But when $\alpha$ is any other integer, $u$ is in general not expressible by means of functions of $\theta$ which give readily the characteristics of the orbits.

It will be shown that by considering equations (3) and (4) the characteristic properties of the orbits for all integral values of $\alpha$ can be readily determined.

It is seen from physical considerations that for attractive forces the orbits are always concave toward the origin, and for repulsive forces always convex. This is evident also from equations (3) since a curve is concave toward the origin if $u+\frac{d^2u}{d\theta^2}>0$, and convex if $u+\frac{d^2u}{d\theta^2}<0$.

It is also to be noticed that when $u$ changes from positive to negative the left member of the first equation of (3) changes sign, and that the right member changes sign if $\alpha$ is odd, but retains its sign if $\alpha$ is even. Hence, for $+m^2$ a negative value of $u$ is allowable for attraction if $\alpha$ is odd, but must be considered as indicating repulsion if $\alpha$ is even. For $-m^2$ a corresponding statement is true. Remembering this, we can write (4) as follows:

$$(6)\qquad \begin{cases} \left(\dfrac{du}{d\theta}\right)^2 = \pm\dfrac{2m^2}{\alpha-1}u^{\alpha-1}-u^2+c\equiv\phi(u), & (\alpha\neq 1), \\ \left(\dfrac{du}{d\theta}\right)^2 = \pm 2m^2\log u-u^2+c\equiv\phi(u), & (\alpha=1). \end{cases}$$

We get from (3) and (6),

$$(7)\qquad dt=\frac{d\theta}{hu^2}=\frac{du}{hu^2\sqrt{\phi(u)}}.$$

Now $\frac{du}{d\theta}=0$ is a necessary and sufficient condition for an apse since, if $\frac{du}{d\theta}=0$, then $\frac{dr}{d\theta}=0$, $(r=\frac{1}{u})$; that is, the orbit is perpendicular to the radius vector. And since $\left(\frac{du}{d\theta}\right)^2\equiv\phi(u)$ the number and values of the roots of $\phi(u)=0$ determine the number and distances of the apses of the orbits.

We can then discuss the orbits from the consideration of the graphs of $\phi(u)$. This is easily obtained by adding the corresponding ordinates of the graphs whose equations are, for $a \neq 1$,

$$y_1 = \pm \frac{2m^2}{a-1} u^{a-1}, \quad y_2 = -u^2, \quad y_3 = c;$$

and for $a=1$,

$$y_1 = \pm 2m^2 \log u, \quad y_2 = -u^2, \quad y_3 = c.$$

It appears in the construction of the graphs that the type of curve is different for different values of $a$, and that it is necessary to consider the following cases: I, $a>3$ and odd; II, $a>3$ and even; III, $a=3$; IV, $a=2$; V, $a=1$; VI, $a<1$ and even; and VII, $a<1$ and odd. It is also found that, except for $a=3$, we get the same types of curves, having the same number of maxima and minima, and giving the same combinations of real positive, real negative, zero, and multiple roots of $\phi(u)=0$, by taking a special value of the parameter $m^2$ and varying the parameter $c$, as by varying both $m^2$ and $c$. There is no loss in the generality of the results if we consider only a single value of $m^2$, with the exceptional case $a=3$, where the type of curve depends upon whether $m^2$ is greater than, equal to, or less than, unity.

The discussion is divided into seven cases, depending upon the value of $a$, and these are subdivided according to the value of $c$. The first case is given somewhat in detail, but in the other cases only the special peculiarities are pointed out. The time required for the maximum variation of $u$ (which is denoted by $T$) is discussed after the other characteristics have been ascertained. Let $\Theta$ denote the angle described by the radius vector for the maximum variation of $u$.

### Case I. $a>3$ and Odd.

The types of graph of $\phi(u)$ for the different values of $c$ are shown in Fig. I, the sub-cases being $(a)$, $(b)$, $(c)$, $(d)$, and $(e)$ for attraction, and $(f)$, $(g)$, and $(h)$ for repulsion. Since $a$ is odd, both positive and negative $u$'s may be used.

*Sub-Case* $(a)$. The characteristics of the orbits in this case are, (1) they have no apse, (2) they extend from the origin to infinity, and (3) $\Theta$ is finite.

It is evident that, since $\phi(u)=0$ has no real roots (as is seen from the graph), there can be no apse. If the initial conditions are $\theta=0$, $u=u_0$, $\frac{du}{d\theta}=\frac{du_0}{d\theta}$, then, since $\left(\frac{du}{d\theta}\right)^2>0$ for every $u$, $\frac{du_0}{d\theta}$ may be either positive or negative, but cannot be zero. If $\frac{du_0}{d\theta}>0$, then as $\theta$ increases $u$ increases

until $\frac{du}{d\theta}$ becomes negative, which can occur only by $\left(\frac{du}{d\theta}\right)$ passing through

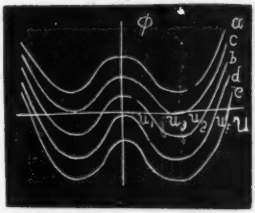



Fig. I. $a>3$ and odd. Attraction.

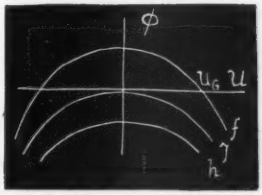



Fig. I. $a>3$ and even. Repulsion.

infinity. Since $\left(\frac{du}{d\theta}\right)^2$ becomes infinite only for $u$ infinite, $u$ increases indefinitely. If $\frac{du_0}{d\theta}<0$, it follows from a similar argument that $u$ decreases to negative infinity. Hence $u$ may vary from $+\infty$ to $-\infty$. That is, $r=1/u$ may vary from 0 to $+\infty$, change to $-\infty$ and return to 0; or the orbit may extend from the origin to infinity and back to the origin.

The proof that $\Theta$ is finite comes from equations (5). Since the denominator of the integrand is never zero, and since $\sqrt{\phi(\infty)}$ is infinite of order greater than unity, $\Theta=\int_0^\infty \frac{du}{\sqrt{\phi(u)}}$ is finite. This means that the orbit makes a finite number of revolutions around the origin between zero and infinity. The type of orbits for this case is shown in Fig. A.

Fig. A.

*Sub-Case* (*b*). The characteristics of the orbits are (1) they have two apse distances, a single orbit, however, having only one apse distance, (2) the orbits lie either entirely within one circle, or entirely without a larger circle with the same center, the first orbit going to the origin, the second to infinity, and (3) $\Theta$ is finite for each orbit.

The real roots of $\phi(u)=0$ are $\pm u_1$, and $\pm u_2$; hence there are apses at $r=1/u_1$ and $r=1/u_2$. But a single orbit can have only one of these apse distances, since $\theta=\int_{u_1}^{u_2} \frac{du}{\sqrt{\phi(u)}}$ is imaginary, giving an imaginary orbit, between these two apse distances. The discussion of the orbits in this case is given in three parts:

(1) If the initial conditions are $\theta=0$, $u=u_0$, $0<|u_0|u_1$, $\frac{du}{d\theta}=\frac{du_0}{d\theta}$; then, if $\frac{du_0}{d\theta}>0$, $u$ increases and, as may be seen from the graph, $\frac{du}{d\theta}$ decreases, becoming zero at $u=u_1$. Now $\frac{d^2u}{d\theta}=\frac{1}{2}\frac{d\phi(u)}{du}$. Therefore, as

is seen from the graph, $\frac{d^2u}{d\theta^2} \neq 0$ at $u=u_1$. Hence $\frac{du}{d\theta}$ continues to decrease and changes sign at $u=u_1$, and $u$ begins to decrease. $\frac{du}{d\theta}$ does not change sign again until $u=-u_1$, where it becomes positive and $u$ again increases to $u=+u_1$. This varying of $u$ between $-u_1$ and $+u_1$ continues indefinitely as $\theta$ increases. If $\frac{du_0}{d\theta}<0$, the same range of variation of $u$ results. We have then for the variation of $r$, $-u_1 \leqq \frac{1}{r} \leqq +u_1$ or $\frac{1}{u_1} \leqq |r| \leqq \infty$. That is, the orbits lie outside the circle whose equation is $r=1/u_1$.

(2) If initially, $u=u_0$, $u_1<|u_0|<u_2$, then $\theta=\int_{u_0}^{u'_0} \frac{du}{\sqrt{\phi(u)}}$, $(u_1<|u'_0|<u_2)$, is imaginary. This means that there is no real orbit for these initial conditions and the given value of $c$.

(3) If the initial conditions are $\theta=0$, $u=u_0$, $u_2<u_0$, $\frac{du}{d\theta}=\frac{du_0}{d\theta}$, then, if $\frac{du_0}{d\theta}>0$, $u$ increases indefinitely. If $\frac{du_0}{d\theta}<0$, $u$ decreases to $u_2$ where $\frac{du}{d\theta}$ becomes zero and changes sign, after which $u$ increases indefinitely. If $u_0<-u_2$ then $|u|$ has the same range of variation as if $u_0>u_2$; namely, $u_2 \leqq |u| \leqq \infty$. Hence we have $0 \leqq |r| \leqq 1/u_2$; that is, the orbits lie inside the circle whose equation is $r=1/u_2$, and have apses on this circle.

Since $\sqrt{[\phi(u)]}$ is an infinitesimal of order less than unity at $u=u_1$, $\Theta=\int_0^{u_1} \frac{du}{\sqrt{\phi(u)}}$ is finite. $\sqrt{[\phi(u)]}$ is also an infinitesimal of order less than unity at $u=u_2$ and is infinite of order greater than unity at $u=\infty$; therefore $\Theta=\int_{u_2}^{\infty} \frac{du}{\sqrt{\phi(n)}}$ is finite.

The type of orbits for this case is shown in Fig. B.

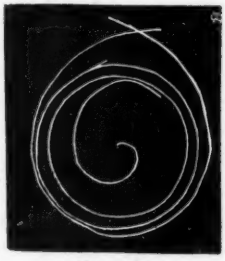


Fig. B.

*Sub-Case* (c). The characteristics of the orbits are (1) they have one apse distance, (2) they either extend from the origin to a certain circle whose center is at the origin, or always move on that circle, or extend from that circle to infinity, and (3), $\Theta$ is infinite for those orbits, which are not circles, the orbits approaching the circles asymptotically both from the inside and from the outside.

The real roots of $\phi(u)=0$ are $\pm u_3$; hence there is an apse at $r=1/u_3$. The discussion of the orbits is again broken up into three parts.

(1) If the initial conditions are $\theta=0$, $u=u_0$, $0 \leqq |u_0| < u_3$,

$\frac{du}{d\theta}=\frac{du_0}{d\theta}$, then, if $\frac{du_0}{d\theta}>0$, $u$ increases to $+u_3$; and if $\frac{du_0}{d\theta}<0$, $u$ decreases to $-u_3$. Since $\phi(u)=0$ has a double root at $u=\pm u_3$, $\frac{1}{\sqrt{\phi(u)}}$ is infinite of the first order at $\pm u_3$. Hence $\Theta_1=\int_{u_0}^{\pm u_3}\frac{du}{\sqrt{\phi(u)}}$ is infinite. That is, the orbit approaches the circle $r=1/u_3$ asymptotically from the outside. Since $\frac{1}{\sqrt{\phi(u)}}$ is finite except at $u=\pm u_3$, the angle described by the radius vector going along the orbit from any point outside the circle to infinity is finite.

(2) If the initial conditions are $\theta=0$, $u=u_0=\pm u_3$, then $\frac{du_0}{d\theta}=0$, and $\frac{d^2u_0}{d\theta^2}=0$. Since $\theta=\int_{u_3}^{u'}\frac{du}{\sqrt{\phi(u)}}$ is infinite for any $u'$ different from $u_3$, the orbit is the circle $r=1/u_3$.

(3) If the initial conditions are $\theta=0$, $u=u_0$, $u_0>u_3$, $\frac{du}{d\theta}=\frac{du_0}{d\theta}$, then if $\frac{du_0}{d\theta}>0$, $u$ increases indefinitely as $\theta$ increases, and $\theta_1=\int_{u_0}^{\infty}\frac{du}{\sqrt{\phi(u)}}$ is finite. If $\frac{du_0}{d\theta}<0$, $u$ decreases to $u_3$, and $\theta_2=\int_{u_0}^{u_3}\frac{du}{\sqrt{\phi(u)}}$ is infinite. That is, $r$ decreases to zero from any point inside the circle $r=1/u_3$ for a finite value of $\theta$, and increases to $1/u_3$ for an infinite value of $\theta$. The orbit approaches asymptotically the circle $r=1/u_3$. The same orbit is obtained for $u_0<-u_3$.

*Sub-Case* (*c*) may be considered as the limiting case of *sub-case* (*b*), as the two circles approach coincidence. From this point of view one would say that the orbits have apses at $r=1/u_3$. It is also a limiting case of *sub-case* (*a*); whence the orbits might be considered as extending from the origin to infinity.

The type of orbits for this case is shown in Fig. C.

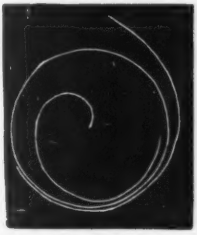


Fig. C.

*Sub-Case* (*d*). The characteristics of the orbits are, (1) they are either circles of infinite radius or finite orbits having one apse distance, (2) the finite orbits go to the origin, and (3) $\Theta$ is finite. The discussion in this case is essentially the same as that in *sub-case* (*b*). The outer orbit has become a circle at infinity.

Fig. D.

*Sub-Case* (*e*). This is the same as the preceding sub-case except that the orbits at infinity have vanished.

*Sub-Case* (*f*). Here we are dealing with a repulsive force. The characteristics of the orbits are,

(1) they have one apse distance, and (2) they go to infinity, having a finite $\Theta$. The discussion is made in the same manner as previously. It is seen that, since the orbit is convex toward the origin, $\Theta < \pi$.

*Sub-Case* (*g*). The orbit has gone to infinity.

*Sub-Case* (*h*). No real orbits exist.

Fig. E.

## Case II. $\alpha > 3$ and Even.

The types of graph of $\phi(u)$ for the different values of $c$ are shown in Fig. II. Since $\alpha$ is even, a negative value of $u$ must be considered as indicating a repulsive force. We get all types of orbits for both attraction and repulsion if we consider only $+m^2$ in the discussion of this case. As $u$ passes through zero, that is, as the orbit passes through infinity, the force changes from attraction to repulsion. The discusssion of the sub-cases involves no new features.

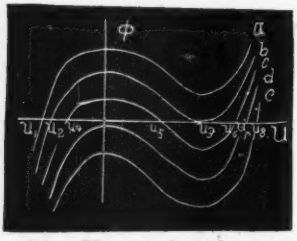



Fig. II. $\alpha > 3$ and even. Attraction and Repulsion.

The characteristics of the orbits in all cases are shown in the Table of Results, in which the italics indicate repulsive forces. The graphs are numbered to correspond to the cases. For $\alpha = 3$ the graph of $\phi(u)$ and the types of orbits for repulsion are the same as in Case I, and these are not repeated. In Case V the graphs of $\phi(u)$ for both attraction and repulsion are shown in the same figure; for all negative values of $u$, $\phi(u)$ is imaginary.

The following are peculiarities of the orbits not already discussed.

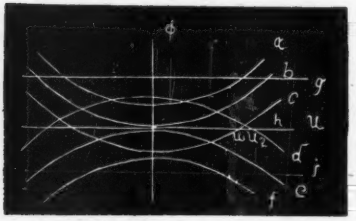



Fig. III. $\alpha = 3$. Attraction.

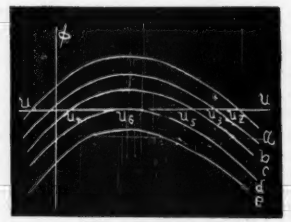



Fig. IV. $\alpha = 2$. Attraction and Repulsion.

(1) In Case III, $\alpha = 3$, there are orbits which go from any finite distance to either the origin or infinity, making an infinite number of revolutions around the origin in each case.

(2) In Case IV, $\alpha=2$ (the Newtonian Law), we find for the first time an orbit having two apse distances.

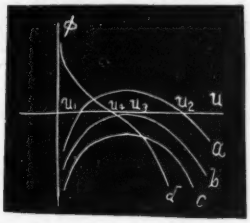



Fig. V. $\alpha=1$.
Attraction and Repulsion.

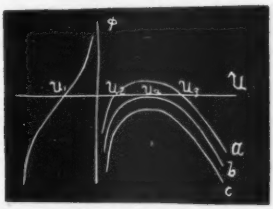



Fig. VI. $\alpha<1$ and even.
Attraction and Repulsion.

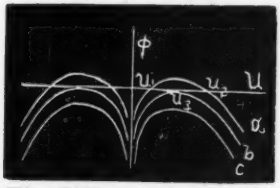



Fig. VII. $\alpha<1$ and odd.
Attraction.

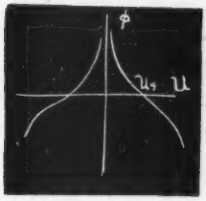



Fig. VII. $\alpha<1$ and odd.
Repulsion.

(3) In Cases V, VI, and VII, $\alpha=2$, the orbits for attraction all have two apse distances, and the orbits lie between the two circles $r=1/u_1$ and $r=1/u_2$, where $u_1$ and $u_2$ are the real roots of $\phi(u)=0$. No orbit goes to the origin or to infinity. Periodic orbits may be possible in these cases.

Fig. F.

(4) In Cases I, II, and III, $\alpha>2$, orbits for attraction lie outside of the region between the two circles $r=1/u_1$ and $r=1/u_2$. In Case IV, $\alpha=2$, there are orbits between two circles and also orbits which go to infinity. The Newtonian Law thus gives an intermediate case between the two general classes of orbits.

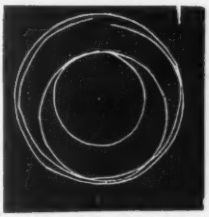


Fig. G.

(5) No orbit in any case has more than two apse distances.

(6) There are orbits which have no apse.

(7) In cases of attraction a circle is a possible orbit for all values of $\alpha$.

*Discussion of T.* From equation (7) we get

$$t_1 - t_0 = \int_{u_0}^{u_1} \frac{du}{hu^2 \sqrt{\phi(u)}} = \int_{u_0}^{u_1} \frac{du}{hu^2 \sqrt{\pm \frac{2m^2}{a-1} u^{a-1} - u^2 + c}}, \quad (a \neq 1).$$

It is now shown under what conditions $t_1 - t_0$ is finite, and these criteria are then applied to the discussion of $T$.

If the limits are finite the integral is finite, if the integrand is infinite of order less than unity at every point of the interval $(u_0\ u_1)$. (1) If the limits do not include $u=0$, the integral is infinite only if $\phi(u)=0$ has a multiple root between or at the limits. (2) If the limits include $u=0$, the integral is infinite unless $1/\sqrt{[\phi(u)]}$ is infinite of order greater than one, when it is in general finite. This is equivalent to saying that the integral is finite at $u=0$ only if $a<-1$. (3) If one limit is $\infty$ and the other so taken that the integrand never becomes infinite between the limits, then the integral is finite if $u^2 1/\sqrt{[\phi(u)]}$ at $u=\infty$ is infinite of order greater than unity. In other words the integral is finite if $a>-1$.

Applying these criteria we find that in Cases I, II, III, IV, and V, for both attraction and repulsion, and in Cases VI and VII for attraction, we can make the general statement that all orbits going to infinity and all those which approach a circle asymptotically are not traversed in a finite time, but for all other orbits $T$ is finite. In Cases VI and VII for repulsion we see from criterion (2) that $T$ is infinite if $a \geqq -1$, but finite if $a<-1$; hence we have the interesting result that, *if a repulsive force varies directly as a power of the distance greater than unity, the particle goes to infinity in a finite time.*

TABLE OF RESULTS.

| Case | Sub-Case | Number of Apses | Variation of $r$ in One Orbit | $\Theta$ | Type of Orbit | $T$ |
|---|---|---|---|---|---|---|
| I. $a>3$ and odd. | (*a*) | none | $0 \leqq \lvert r \rvert \leqq \infty$ | finite | Fig. A | $\infty$ at $r=\infty$ |
| | (*b*) | two | $0 \leqq \lvert r \rvert \leqq 1/u_2$<br>$1/u_1 \leqq \lvert r \rvert \leqq \infty$ | finite<br>finite | Fig. B | finite<br>$\infty$ at $r=\infty$ |
| | (*c*) | one | $0 \leqq \lvert r \rvert \leqq 1/u_3$<br>$1/u_3 \leqq \lvert r \rvert \leqq \infty$ | $\infty$ at $r=1/u_3$ | Fig. C | $\infty$ at $r=1/u_3$<br>$\infty$ at $r=\infty$ |
| | | | $r=1/u_3$ | | circle | |
| | (*d*) | one | $r=\infty$ | | | |
| | | | $0 \leqq \lvert r \rvert \leqq 1/u_4$ | finite | Fig. D | finite |
| | (*e*) | one | $0 \leqq \lvert r \rvert \leqq 1/u_5$ | finite | Fig. D | finite |
| | (*f*) | *one* | $1/u_6 \leqq \lvert r \rvert \leqq \infty$ | *finite* | *Fig. E* | $\infty$ at $r=\infty$ |
| | (*g*) | | $r=\infty$ | | | |
| | (*h*) | | *none* | | | |
| II. $a>3$ and even. | (*a*) | one | $0 \leqq \lvert r \rvert \leqq \infty$ | finite | Fig. A | $\infty$ at $r=\infty$ |
| | | | $1/u_1 \leqq \lvert r \rvert \leqq \infty$ | *finite* | *Fig. E* | $\infty$ at $r=\infty$ |
| | (*b*) | two | $1/u_3 \leqq \lvert r \rvert \leqq \infty$<br>$0 \leqq \lvert r \rvert \leqq 1/u_3$ | $\infty$<br>$\infty$ | Fig. C | $\infty$ at $r=1/u_3$<br>$\infty$ at $r=\infty$ |
| | | | $r=1/u_3$ | | circle | |
| | | | $1/u_2 \leqq \lvert r \rvert \leqq \infty$ | *finite* | *Fig. E* | $\infty$ at $r=\infty$ |
| | (*c*) | three | $1/u_5 \leqq \lvert r \rvert \leqq \infty$<br>$0 \leqq \lvert r \rvert \leqq 1/u_6$ | finite<br>finite | Fig. B | $\infty$ at $r=\infty$<br>finite |
| | | | $1/u_4 \leqq \lvert r \rvert \leqq \infty$ | *finite* | *Fig. C* | $\infty$ at $r=\infty$ |
| | (*d*) | two | $r=\infty$ | | | |
| | | | $0 \leqq \lvert r \rvert \leqq 1/u_7$ | finite | Fig. D | finite |
| | (*e*) | one | $0 \leqq \lvert r \rvert \leqq 1/u_8$ | finite | Fig. D | finite |

| Case | Sub-Case | Number of Apses | Variation of $r$ in One Orbit | $\Theta$ | Type of Orbit | $T$ |
|---|---|---|---|---|---|---|
| III. $a=3$. | $m^2>1$ (a) | none | $0 \leqq \mid r \mid \leqq \infty$ | $\infty$ at $r=0$ | Fig. A | $\infty$ at $r=\infty$ |
| | $m^2>1$ (b) | one | $0 \leqq \mid r \mid \leqq \infty$ | $\infty$ at $r=0$, $r=\infty$ | Fig. A | $\infty$ at $r=\infty$ |
| | $m^2>1$ (c) | one | $0 \leqq \mid r \mid \leqq 1/u_1$ | $\infty$ at $r=0$ | Fig. D | finite |
| | $m^2<1$ (d) | one | $1/u_2 \leqq \mid r \mid \leqq \infty$ | finite | Fig. F | $\infty$ at $r=\infty$ |
| | $m^2<1$ (e) | one | $r=\infty$ | | | |
| | $m^2<1$ (f) | none | | | | |
| | $m^2=1$ (g) | none | $0 \leqq \mid r \mid \leqq \infty$ | $\infty$ at $r=0$ | Fig. A | $\infty$ at $r=\infty$ |
| | $m^2=1$ (h) | $\infty$ | $r=c$, $0 \leqq c \leqq \infty$ | | circle | |
| | $m^2=1$ (i) | none | | | | |
| | | | For repulsion, see Case I, Repulsion. | | | |
| IV. $a=2$. | (a) | two | $1/u_2 \leqq \mid r \mid \leqq \infty$ | finite | Fig. F | $\infty$ at $r=\infty$ |
| | | | $1/u_1 \leqq \mid r \mid \leqq \infty$ | *finite* | *Fig. E* | $\infty$ at $r=\infty$ |
| | (b) | two | $1/u_3 \leqq \mid r \mid \leqq \infty$ | finite | Fig. F | $\infty$ at $r=\infty$ |
| | (c) | two | $1/u_5 \leqq \mid r \mid \leqq 1/u_4$ | finite | Fig. G | finite |
| | (d) | one | $r=1/u_6$ | | circle | |
| | (e) | | none | | | |
| V. $a=1$. | (a) | two | $1/u_2 \leqq \mid r \mid \leqq 1/u_1$ | finite | Fig. G | finite |
| | (b) | one | $r=1/u_3$ | | circle | |
| | (c) | | none | | | |
| | (d) | *one* | $1/u_4 \leqq \mid r \mid \leqq \infty$ | *finite* | *Fig. E* | $\infty$ at $r=\infty$ |
| VI. $a<1$ and even. | (a) | three | $1/u_3 \leqq \mid r \mid \leqq 1/u_2$ | finite | Fig. G | finite |
| | | | $1/u_3 \leqq \mid r \mid \leqq \infty$ | *finite* | *Fig. E* | *finite*, $a<-1$; $\infty$, $a \geqq -1$ |
| | (b) | two | $r=1/u_4$ | | circle | |
| | | | $1/u_1 \leqq \mid r \mid \leqq \infty$ | *finite* | *Fig. E* | *finite*, $a<-1$; $\infty$, $a \geqq -1$ |
| | (c) | one | $1/u_1 \leqq \mid r \mid \leqq \infty$ | *finite* | *Fig. E* | *finite*, $a<-1$; $\infty$, $a \geqq -1$ |
| VII. $a<1$ and odd. | (a) | two | $1/u_2 \leqq \mid r \mid \leqq 1/u_1$ | finite | Fig. G. | finite |
| | (b) | one | $r=1/u_3$ | | circle | |
| | (c) | | none | | | |
| | (d) | *one* | $1/u_4 \leqq \mid r \mid \leqq \infty$ | *finite* | *Fig. E* | *finite*, $a<-1$; $\infty$, $a \geqq -1$ |

# DEPARTMENTS.

## SOLUTIONS OF PROBLEMS.

### ALGEBRA.

**291. Proposed by L. E. NEWCOMB, Los Gatos, Cal.**

An empty water tank has two inflow pipes $A$, $B$, which begin to flow at the same moment. When $B$, the smaller pipe, has discharged $s$ gallons, and the tank is $1/n$ filled, water from both pipes is turned off. After $A$, $B$, have been idle, each as many hours as would suffice it to perform $1/m$ the work done previously by the other pipe, the flow, which is of a uniform rate, is resumed and continued till the tank is filled; $B$ during the second working period has discharged $t$ gallons. (1) What is the capacity of the tank? (2) What would be the capacity if $B$ were an outflow pipe?

**Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.**

Let $C$=capacity, $x$=number of hours for $A$ to fill tank, $y$=number of hours for $B$ to fill tank. Then $C/x$=what $A$ does in one hour, $C/y$=what $B$ does in one hour. $sy/C$=time for $B$ to discharge $s$ gallons, $(C-sn)x/(Cn)$ =time for $A$ to discharge $C/n-s$ gallons, $sx/Cm$=time $A$ is idle, $(C-sn)y/(Cmn)$=time $B$ was idle, $(s+t)y/C$=total time $B$ works, and $(C-s-t)x/C$ =total time $A$ works.

But $$\frac{(s+t)y}{C}=\frac{(C-s-t)x}{C}-\left(\frac{(C-sn)y}{Cmn}-\frac{sx}{Cm}\right).$$

$$\therefore \left(\frac{s+t}{C}+\frac{C-sn}{Cmn}\right)y=\left(\frac{C-s-t}{C}+\frac{s}{Cm}\right)x \ldots (1).$$

Also $$\frac{sy}{C}=\frac{(C-sn)x}{Cn}, \text{ or } y=\frac{(C-sn)x}{sn} \ldots (2).$$

(2) in (1) gives $C=mn(sn-s-t)+2sn$.

II. $\frac{(C+sn)x}{Cn}$=time $A$ works before turned off.

$\frac{(C+s+t)x}{C}$=total time $A$ works, $\frac{(C+sn)y}{Cmn}$=time $B$ was idle.

$$\therefore \left(\frac{s+t}{C}+\frac{C+sn}{Cmn}\right)y=\left(\frac{C+s+t}{C}+\frac{s}{Cm}\right)x\ldots(3);\ y=\frac{(C+sn)x}{sn}\ldots(4).$$

(4) in (3) gives $C=mn(sn-s-t)-2sn$.

**292. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.**

Find the sum of the series $1^2+5^2+14^2+30^2+\ldots+[\frac{1}{6}n(n+1)(2n+1)]^2$.

**Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.**

The differences and the terms of this special series may be arranged as follows for the first seven terms:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $u_1=$ | $1^2$ | | $5^2$ | | $14^2$ | | $30^2$ | | $55^2$ | | $91^2$ | | $140^2$ |
| $u_1=$ | 1 | | 25 | | 196 | | 900 | | 3025 | | 8281 | | 19600 |
| $\Delta^1u_1=$ | . | 24 | | 171 | | 704 | | 2125 | | 5256 | | 11319 | . |
| $\Delta^2u_1=$ | . | . | 147 | | 533 | | 1421 | | 3131 | | 6063 | . | . |
| $\Delta^3u_1=$ | . | . | . | 386 | | 888 | | 1710 | | 2932 | . | . | . |
| $\Delta^4u_1=$ | . | . | . | . | 502 | | 822 | | 1222 | . | . | . | . |
| $\Delta^5u_1=$ | . | . | . | . | . | 320 | | 400 | . | . | . | . | . |
| $\Delta^6u_1=$ | . | . | . | . | . | . | 80 | . | . | . | . | . | . |

Compute the series for ten terms, or more, and it will be found that $\Delta^6u_1$ are all 80, or constant, therefore all the higher differences vanish. To sum the series we have the value of the leading term and the six leading differences. I have given a general formula for $S_n$, on page 163, of THE AMERICAN MATHEMATICAL MONTHLY for August-September, 1906, see equation $(E)$. We have:

$$S_n=nu_1+\frac{n(n-1)}{2}\Delta^1u_1+\frac{n(n-1)(n-2)}{3!}\Delta^2u_1+\ldots$$

$$+\frac{n(n-1)\ldots(n-6)}{7!}n^6u_1\ldots(1).$$

From the problem and the above table we have: $u_1=1$, $\Delta'=24$, $\Delta^2=147$, $\Delta^3=386$, $\Delta^4=502$, $\Delta^5=320$, and $\Delta^6=80$. Substitute numerical values in (1), expand the terms, consolidate like terms, reduce, and we have:

$$S_n=\frac{20n^7+140n^6+371n^5+455n^4+245n^3+35n^2-6n}{1260}\ldots(2),$$

$$=\tfrac{1}{1260}[n(n+1)(n+2)(2n+1)(2n+3)(5n^2+10n-1)].$$

Also solved by E. B. Escott, and G. B. M. Zerr. Professor Escott solved the problem by putting the general term equal to $A+Bn+Cn(n+1)+...+Gn(n+1)(n+2)(n+3)(n+4)(n+5)$. Then by letting $n=0, -1, -2$, etc., he determines $A, B, ..., G$. The general term is thus reduced to five terms of the form $n(n+1)...(n+r-1)$. Since the sum of a series whose general term is $n(n+1)(n+2)...(n+r-1)$ is $[n(n+1)...(n+r-1)]/[r+1]$ finds the sum which agrees with that obtained by Mr. DeLand.

Dr. Zerr decomposed the general term in a similar way and after summing the five similar series thus arising he gets the same result as that given above.

## GEOMETRY.

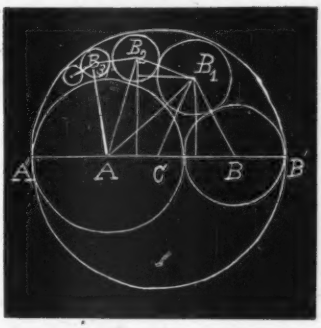
**326. Proposed by L. E. NEWCOMB, Los Gatos, Cal.**

The circle $C$ of radius $pR$ encloses the circles $A_1$, $B_1$ of radii $R$ and $(p-1)R$, respectively; the circle $B_1$ is tangent to $A_1$, $B_1$, $C_1$; the circle $B_2$ is tangent to $A$, $B_1$, $C$; the circle $B_3$ to $A$, $B_2$, $C$, ..., $B_n$ to $A$, $B_{n-1}$, $C$. Find the radius of the circle $B_n$.

**Solution by the PROPOSER.**

First find the locus of centers of circles tangent to $A$ and $C$, taking $A'$ the point of contact of $A$ and $C$ as the origin.

Let $r$, $r_1$, $r_2$, ..., $r_n$ be the radii of $B$, $B_1$, ..., $B_n$, respectively; $r'=$radius of any circle tangent to circles whose centers are $A$, $C$; and $x$, $y$ co-ordinates of its centers. Then $(r'+R)^2-(R-x')^2=(pR-r')^2-(pR-x')^2=y'^2$... (1).




$$\therefore r=\frac{(p-1)x'}{p+1}\text{... (2), and } x'=\frac{(p+1)r'}{p-1}\text{... (3).}$$

Substituting the value of $x'$ in (1), we have

$$(R+r')^2-(R-\frac{p+1}{p-1}r')^2=y'^2.$$

$$\therefore y'=\frac{2}{p-1}\sqrt{[p(p-1)Rr'-pr'^2]}.$$ Since $r=(p-1)R$ and $x=R(p+1)$, $(p-1)R$ is of the form $\frac{p(p-1)R}{0^2(p-1)^2+p}$.

2. Find $r_1$. Join centers of $A$, $B$, and $C$ with $B_1$, $B_2$, ..., $B_n$. Draw perpendiculars from centers $B_1$, $B_2$, ..., to the diameter of $C$ passing through $A'$.

$$(r+r'_1)^2-(x-x_1)^2=(pR-r_1)^2-(pR-x_1)^2\text{... (4).}\quad x_1=\left(\frac{p+1}{p-1}\right)r_1.$$

Substitute the values of $x$, $r$, $x_1$ in (4); whence

$$4Rr_1(p^2-p+1)=(p-1)pR^2.\quad \therefore r_1=\frac{p(p-1)R}{(p-1)^2+p}=\frac{p(p-1)R}{1^2(p-1)^2+p}.$$

(3). Find $r_2$. Let $(p-1)=s$. $(r_1+r_2)^2-(x_1-x_2)^2=(y_1-y_2)^2$..(5).

$$r_1^2=\left(\frac{ps}{s^2+p}\right)^2R^2;\ x_1^2=\frac{p^2(p+1)^2R^2}{(s^2+p)^2};\ x_2^2=\frac{(p+1)^2}{s^2}r_2^2;\ y_1=\frac{2pRs^2}{s(s^2+p)};$$

$$y_2=\frac{2}{s}\sqrt{[pRsr_2-pr_2^2]}.$$

Equation (5) becomes, after taking square root of both members,

$$\sqrt{\frac{(ps)^2R^2}{s^2+p}-\frac{p^2(p+1)^2R^2}{(s^2+p)^2}+\frac{2p(p+1)^2r_2R}{s(s^2+p)}+r_2^2-\frac{(p+1)^2r_2^2}{s^2}}$$

$$=\frac{2pRs^2}{s(s^2+p)}-\frac{2}{s}\sqrt{[pRsr_2-pr_2^2]}\ldots(6).$$

After clearing of radicals and reducing, (6) becomes

$$\frac{p[(s^2-p^2)+4s^2p]R^2}{(s^2+p)^2}+\frac{p+4s^2}{s^2}r_2^2-\frac{2[s^2(2p^2-p+2)+p^2]Rr_2}{s(s^2+p)}=0\ldots(7).$$

$$\therefore r_2=\frac{s[s^2(2p^2-p+2)+p^2]R}{(s^2+p)(p+4s^2)}$$

$$-\frac{R\sqrt{\{[s^2(2p^2-p+2)+p^2]^2-p(p+4s^2)(s^2+p)^2\}}}{(s^2+p)(p+4s^2)},$$

$$R\sqrt{\{[s^2(2p^2-p+2)+p^2]^2-p(p+4s^2)(s^2+p)^2\}}$$

$$=2s^2(s^2+p)=s^2(2p^2-2p+2).$$

$$\therefore r_2=\frac{Rsp(s^2+p)}{(s^2+p)(p+4s^2)}=\frac{p(p-1)R}{2^2(p-1)^2+p}.$$

3. Find $r_n$. Assume that $r_{n-1}=\dfrac{p(p-1)R}{(n-1)^2(p-1)^2+p}$.

Let $(n-1)=t$; then $r_{n-1}=\dfrac{pRs}{t^2s^2+p}$. $(r_t+r_n)^2-(x_t-x_n)^2=(y_t-y_n)^2\ldots(8)$.

$$x_t=\frac{p(p+1)R}{t^2s^2+p};\ y_t=\frac{2tpsR}{t^2s^2+p};\ y_n=\frac{2}{s}\sqrt{[psRr_n-pr_n^2]}.$$

As in case 2, (8) becomes

$$\frac{[t^2s^2+p-(p^2+1)]^2+4t^2ps^2}{(s^2t^2+p)^2}r_n^2-\frac{2ps\,[(t^2s^2-p)t^2s^2+p-(p^2+1)]Rr_n}{s^2t^2+p}$$

$$+\frac{4t^2(p^2+1)s^2ps}{s^2t^2+p}Rr_n+\frac{p^2s^2[(t^2s^2-p)^2+4t^2ps^2]R^2}{(s^2t^2+p)^2}=0\ldots(9);\ \text{whence}$$

$$r_n=\frac{psR\{(t^2s^2-p)[t^2s^2+p-(p^2+1)]+2t^2s^2(p^2+1)\}}{(t^2s^2+p)[t^2s^2+p-(p^2+1)^2+4t^2ps^2}$$

$$-[pRs\sqrt{\{(t^2s^2-p[t^2s^2+p-(p^2+1)+2t^2s^2(p^2+1)]^2}$$
$$-(t^2s^2+p)^2[t^2s^2+p-(p^2+1)]^2+4pt^2s^2)\}]$$
$$/(t^2s^2+p)[t^2s^2+p-(p^2+1)]^2+4t^2ps.$$

The quantity under the radical$=2t^3s^4+2tps^2$.

$$\therefore r_n=\frac{pRs\{[(t^2s^2-p)(t^2s^2+p-(p^2+1)]+2t^2s^2(p^2+1)-2ts^2(s^2t^2+p)\}}{(t^2s^2+p)[t^2s^2+p-(p^2+1)]^2+4ps^2t^2}$$

$$=[pRs\{(t^2s^2-p)[(t^2s^2+p-(p^2+1)]+2t^2s^2(p^2+1)]-2ts^2(s^2t^2+p)\}]$$
$$/\{(t^2s^2-p)[(t^2s^2+p-(p^2+1)]+2t^2s^2(p^2+1)$$
$$-2ts^2(s^2t^2+p)\}\{(t+1)^2(p-1)^2+p\}.$$

$$=\frac{psR}{(t+1)^2(p-1)^2+p}=\frac{p(p-1)R}{n^2(p-1)^2+p}.$$

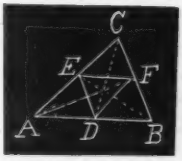
Since it has been shown that this expression is true for $B_1$ and $B_2$, it follows that it is true for $B_n$.

Excellent demonstrations were received from G. B. M. Zerr and C. E. White.

**327. Proposed by J. C. CORBIN, Pine Bluff, Ark.**

In triangle $ABC$, the triangle $DEF$ is formed by joining the feet of the medians and four parallelograms are also formed, viz., $AEFD$, $BFED$, and $CEDF$. Let $a$, $b$, $c$; $d$, $e$, $f$ represent the three medians of $ABC$, and the three sides of $DEF$. Then the sum of the squares of the six diagonals equals the sum of the squares of the twelve sides of the parallelograms, which are equal in sets of four. That is, $a^2+b^2+c^2+d^2+e^2+f^2=4(d^2+e^2+f^2)$, or $a^2+b^2+c^2=3(d^2+e^2+f^2)=3/4(AB^2+BC^2+CA^2)$.

**Solution by J. H. MEYER, S. J., Augusta, Ga.**




Let $CD=a$, $AF=b$, $EB=c$, $DF=e$, $EF=d$, and $ED=f$. Now, by geometry, we know that

$a^2+d^2$ in parallelogram $ECFD=2f^2+2a^2$;
$b^2+f^2$ in parallelogram $AEFD=2e^2+2d^2$;
$c^2+e^2$ in parallelogram $BFED=2f^2+2d^2$.

$\therefore a^2+b^2+c^2+d^2+e^2+f^2=4[e^2+f^2+d^2]$;
$\therefore a^2+b^2+c^2=3[e^2+f^2+d^2]$.

Since $e^2+f^2+d^2$ (by construction) $=\frac{AC^2}{4}+\frac{CB^2}{4}+\frac{AB^2}{4}$,

$\therefore a^2+b^2+c^2=\frac{3}{4}[AC^2+CB^2+AB^2]$.

Also solved by G. B. M. Zerr, J. Scheffer, and the Proposer.

## CALCULUS.

255. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Find the general values of $u$ and $v$ in terms of $x$, which satisfy the equations $u^2+l^2(du/dx)^2=v^2$, $u^2+m^2(du/dx)^2=v^2+n^2(dv/dx)^2$.

Solution by A. F. CARPENTER, Professor of Mathematics, Hastings, Neb.; GEORGE W. HARTWELL, Columbia University, and the PROPOSER.

Subtracting the first equation from the second, we get

$$[m^2-l^2]/n^2[du/dx]^2=[dv/dx]^2.$$

$$\therefore v=\left(\frac{m^2-l^2}{n^2}\right)^{\frac{1}{2}}u+C\ldots(1).$$

$$\therefore u^2+l^2[du/dx]^2=\frac{m^2-l^2}{n^2}u^2+2uC\left(\frac{m^2-l^2}{n^2}\right)^{\frac{1}{2}}+C^2.$$

Let $[m^2-l^2-n^2]/n^2=a$, and $C\left(\frac{m^2-l^2}{n^2}\right)^{\frac{1}{2}}=b$.

Then $l^2[du/dx]^2=au^2+2bu+C^2$.

$$\therefore x=l\int\frac{du}{\sqrt{[au^2+2bu+C^2]}}=\frac{l}{2C}\log\left(\frac{au+b-C}{au+b+C}\right)+\log C_1.$$

$$\therefore x=\frac{l}{2C}\log\left(\frac{C_1[au+b-C}{au+b+C}\right).\quad\text{Let } e^{2Cx/l}=c.$$

Then $c/C_1=\frac{au+b-C}{au+b+C}$, or $u=\frac{[b+C]c-[b-C]C_1}{a[C_1-c]}$.

$$\therefore u=\frac{\{C[(m^2-l^2)^{\frac{1}{2}}+n]/n\}e^{2Cx/l}-C_1C[(m^2-l^2)^{\frac{1}{2}}-n]/n]}{[m^2-l^2-n^2][C_1-e^{2Cx/l}]/n^2}$$

$$=\frac{Cn\{[(m^2-l^2)^{\frac{1}{2}}+n]e^{2Cx/l}-C_1[(m^2-l^2)^{\frac{1}{2}}-n]\}}{[m^2-l^2-n^2][C_1-e^{2Cx/l}]}.$$

$v$ is found at once from (1).

Also solved by V. M. Spunar, J. Scheffer, A. H. Holmes, and J. I. Wodo.

256. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\sum_{x=0}^{x=\infty} \frac{x^{2m}}{1+x^{2n}} = \frac{\pi}{2n\sin\frac{2m+1}{2n}\pi}$, $m$ and $n$ being positive integers of which $n$ is the greater.

I. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$$\sum_{x=0}^{x=\infty} \frac{x^{2m}}{1+x^{2n}} = \int_0^\infty \frac{x^{2m}dx}{1+x^{2n}}.$$

Let $x^{2n}=y$, and let $\frac{2m+1}{2n}=r$. Then

$$\int_0^\infty \frac{x^{2m}dx}{1+x^{2n}} = \frac{1}{2n}\int_0^\infty \frac{y^{r-1}dy}{1+y} = \frac{1}{2n}\Gamma(r)\Gamma(1-r) = \frac{\pi}{2n\sin r\pi} = \frac{\pi}{2n\sin\frac{2m+1}{2n}\pi}.$$

II. Solution by V. M. SPUNAR, Mechanical and Civil Engineer, East Pittsburg, Pa.

This problem belongs in a restricted sense to Algebra; as, however, the process is intimately connected with integration, let $a_1, a_2, a_3, \ldots, a_n$ denote the roots of $x^{2n}+1=0$, the roots being real and unequal; hence, by the theory of equations, $a$ is of the form

$$a=\cos\frac{(2k+1)\pi}{2n}+i\sin\frac{(2k+1)\pi}{2n},$$

in which $k$ is either 0 or a positive integer less than $n$ [$k=0, 1, \ldots, (n-1)$]. Again, by the method of decomposition, we have

$$A-Bi=\frac{a^{2m}}{2n\,a^{2n-1}}=-\frac{a^{2m+1}}{2n}.$$

$\therefore a^{2m+1}=\cos(2k+1)\phi+i\sin(2k+1)\phi$, where $\phi=\frac{(2m+1)\pi}{2n}$.

Hence, $B=\frac{\sin(2k+1)\phi}{2n}$.

$$\therefore B_1+B_2+B_3+\ldots+B_n=\frac{1}{2n}[\sin\phi+\sin3\phi+\ldots+\sin(2n-1)\phi]$$

$$=\frac{1}{2n}\left(\frac{1}{\sin\frac{(2m+1)\pi}{2n}}\right), \text{ by a well known formula in Trigonometry.}$$

$(B_1+B_2+B_3+\ldots+B_n)2\pi=\int_{-\infty}^{\infty}\frac{x^{2m}dx}{1+x^{2n}}$, by a well known formula in Calculus.

Hence, $\sum_{x=-\infty}^{x=+\infty}\frac{x^{2m}}{1+x^{2n}}=\frac{\pi}{n\sin\frac{(2m+1)\pi}{2n}}=2\sum_{x=-0}^{x=+\infty}\frac{x^{2m}}{1+x^{2n}}$.

Hence, $\sum_{x=0}^{x=+\infty}\frac{x^{2m}}{1+x^{2n}}=\frac{\pi}{2n\sin\frac{(2m+1)\pi}{2n}}$.

Also solved by J. I. Wodo.

---

## NUMBER THEORY AND DIOPHANTINE ANALYSIS.

120. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Find the prime numbers $p$ for which $x^2-pxz-px-z+p^2-3=0$ has more than two sets of positive integral solutions $x$, $z$, each $<p$.

II. Solution by E. B. ESCOTT, Ann Arbor, Mich.

The solution of this problem which was published in the March number of THE AMERICAN MATHEMETICAL MONTHLY does not seem to answer the question exactly, but rather proposes instead another problem which seems equally difficult, as there is no certainty that the numbers $p$ so found will be prime. Thus, the solution given leads to the following values of $p$, 74 and 394199, neither of which is prime.

The following method will give an indefinite number of primes $p$ which satisfy the given equation, and leads to formulas such as $p=x^2-x-1$;

$$x=n^2+n-1,\quad p=n^3+n^2-2n-1. \tag{1}$$

In the equation, $$z+1=n=\frac{x^2+p^2-2}{px+1}, \tag{2}$$

$p$ and $x$ occur symmetrically, and $n$ will be unchanged if $p$ and $x$ are interchanged. If the roots of the equation (2), considered as an equation in $x$, are $x_1$ and $x_2$, we have $x_2=p_1n-x_1$, and since, as just noticed, $p$ and $x$ may be interchanged, we may take this value of $x_2$ for a new $p$ (say $p_2$) and take the new $x_1=p_1$. $n$ will be unchanged. The new $x_2=p_2n-x_1=p_2n-p_1$. Let this equal $p_3$, etc. Then we see that in the recurring series

$P_n : x_1,\ p_1,\ p_1n-x_1,\ \ldots$, where the scale of relation is $P_{n+2}=nP_{n+1}-P_n$,

any two adjacent terms of the series may be taken as $x$ and $p$, the initial terms $x_1$ and $p_1$ being any solution of the equation (2).

Example. Let $x_1=1$, $p_1=a$, then $n=a-1$. The series is

$$1,\ a,\ a^2-a-1,\ a^3-2a^2-a+1,\ a^4-3a^3+3a,\ a^5-4a^4+2a^3+5a^2-2a-1.$$

If we take the second and third terms for $x$ and $p$, we shall have the first of the formulas given; with the third and fourth terms, we shall have the second formulas (1).

*Numerical Examples.*

| | |
|---|---|
| $n=2$ | 1, 3, 5, 7, 9, 11, 13, ... |
| $n=3$ | 1, 4, 11, 29, 76, 199, 521, ... |
| $n=4$ | 1, 5, 19, 71, 265, 989, 3691, ... |
| $n=5$ | 1, 6, 29, 139, 666, ... |
| $n=6$ | 1, 7, 41, 239, 1393, ... |

Every prime occurring in this table is a value of $p$ and the term preceding it is the corresponding value of $x$. Examples of primes $p$ with four values of $x$: $p=29$, $x=1, 6, 11, 27$; $p=71$, $x=1, 9, 19, 69$; $p=239$, $x=1, 16, 41, 237$.

**III. Solution by the PROPOSER.**

The following method, which is that developed by the Proposer at the time of setting the problem, will be shown to lead to all possible solutions. If $(p^2-1)^2$ has the factor $1+xp$, the complementary factor has the form $1+yp$ and there exists an integer $k \geq 1$ (designated $z+1$ in the proposed congruence) such that

$$x+y=pk,\ xy+k=p^2-2.$$

Eliminating $y$ and $x$ in turn, we get

$$k=\frac{x^2+p^2-2}{1+xp},\ k=\frac{y^2+p^2-2}{1+yp}\ldots(1),$$

$$(p^2-1)^2=(1+xp)[1+(pk-x)p]=(1+yp)[1+(pk-y)p]\ldots(2).$$

In $(1_1)$, $x$ and $p$ enter symmetrically; in $(1_2)$, $y$ and $p$. Hence

$$(x^2-1)^2=(1+px)[1+(xk-p)x],\quad (y^2-1)^2=(1+py)[1+(yk-p)y]\ldots(3).$$

If $k=1$ in (1), then from $(x-p)^2 \geq 0$ we get $p \leq 2$. We assume that $p \geq 2$, whence $k>1$.

Denote by the symbol $\{x, p\}$ a pair of numbers such that

($i$) $x$ and $p$ are positive integers;

($ii$) $x<p$;

($iii$) $1+xp$ is a factor of $(p^2-1)^2$.

From one such pair we may derive a right-neighboring pair $\{p, y\}$, where $y=pk-x$. Since $x<p$, $k>1$, we have $p<y$, so that properties ($i$) and ($ii$) hold for $\{p, y\}$. Property ($iii$) is true by $(3_2)$. Hence any pair leads to a chain of right-neighboring pairs:

$$\{x, p\},\ \{p, pk-x\},\ \{pk-x,\ (pk-x)k-p\}\ldots(4).$$

For $x=1$, $k=p-1$, by (1). Hence we have the successive pairs

$$\{1, k+1\},\ \{k+1,\ k^2+k+1\},\ \{k^2+k-1,\ k^3+k^2-2k-1\}\ldots(5).$$

When the second member of a pair (5) is a prime $p$, the first member is a solution $x$ ($x<p$). We proceed to show that every pair $\{X, P\}$ leading to a solution and having $P$ prime may be obtained from the pairs (5) by assigning a suitable value to $k$. To this end we consider the pair $\{w, x\}$, $w=xk-p$, which is left-neighboring to a given pair $\{x, p\}$. We determine the conditions under which $\{w, x\}$ has the properties ($i$)—($iii$). The third property holds in view of $(3_1)$. By the latter,

$$wx=A-1,\ A\equiv\frac{(x^2-1)^2}{1+px},\ w=\frac{x^3-2x-p}{1+px}<\frac{x^3}{x^2},$$

since $x<p$. Hence $w<x$, and property ($ii$) holds. Finally, ($i$) holds, viz., $w$ is positive, unless $A=0$ or 1. Hence we may form in succession left-neighboring pairs until we reach a pair with $A=0$ or 1. One of the latter cases must ultimately present itself, since a series of decreasing positive integers must terminate. If $A=0$, then $x=1$, and the chain contains the first pair (5), so that $\{X, P\}$ occurs in the list (5). If $A=1$, then $p=xk$, $k=x^2-2$. The first terms of the pairs (4) are then

$$x,\ x(x^2-2),\ x(x^2-2)^2-x,\ x(x^2-2)^3-2x(x^2-2),\ \ldots$$

a series of increasing integers with the factor $x$, so that no one is a prime $P$.

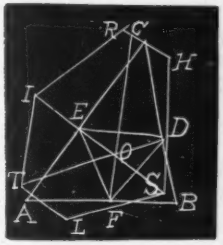
## MECHANICS.

**210. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.**

A rigid triangle is formed of three weightless, smoothly jointed, rigid rods $BC$, $CA$, $AB$. At their mid points $D$, $E$, $F$, respectively, are small, smooth rings, through which passes an endless, stretched, elastic string, forming the triangle $DEF$. Find by graphical construction the reaction at the joints.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let $ABC$ be the triangle, $DEF$ the stretched string, $P$ the force exerted by the string, $O$ the in-center of $DEF$. Then the resultant of the forces in $DE$, $DF$ is $2P\cos\frac{1}{2}A$, acting from $D$ through $O$. Let $OT$ represent the this force. The resultant of the forces in $FE$, $FD$ is $2P\cos\frac{1}{2}C$ acting from $F$ through $O$. Let $OR$ represent this force. The resultant of the forces in $EF$, $ED$ is $2P\cos\frac{1}{2}B$ acting from $E$ through $O$. Let $OS$ represent this force. $OT$ can be replaced by two parallel forces equal to $\frac{1}{2}OT$ acting at $B$ and $C$, respectively; $OR$, by two parallel forces equal to $\frac{1}{2}OR$ acting at $A$ and $B$, respectively; and $OS$, by two parallel forces equal to $\frac{1}{2}OS$ acting at $A$ and $C$, respectively. Completing the parallelograms, we get $OH$ the resultant of $OR$ and $OS$, and $\frac{1}{2}OH$ represents the reaction at $A$. $OI$ is the resultant of $OR$ and $OT$ while $\frac{1}{2}OI$ represents the reaction at $B$. $OL$ is the resultant of $OT$ and $OS$, and $\frac{1}{2}OL$ represents the reaction at $C$.




## AVERAGE AND PROBABILITY.

194. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

What is the mean value of the triangle formed by joining three points taken at random on the circumference of a circle?

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let $OC$=diameter of given circle$=2a$. Let the point $P$ be fixed, draw $PO$ perpendicular to $OC$, and draw $OA$ and $OB$. Let $\angle POA=\theta>\frac{1}{2}\pi$, $\angle POB=\phi<\frac{1}{2}\pi$.

$\therefore OA=2a\sin\theta$, and $OB=2a\sin\phi$. Area $OAB=2a^2\sin\theta\sin\phi\sin(\theta-\phi)$. Average area$=\triangle$,

$$=\frac{\int_0^{\pi}\int_0^{\theta} 2a^2\sin\theta\sin\phi\sin(\theta-\phi)\,d\theta\,d\phi}{\int_0^{\pi}\int_0^{\theta} d\theta\,d\phi}$$

$$=\frac{4a^2}{\pi^2}\int_0^{\pi}\int_0^{\theta}\sin\theta\sin\phi\sin(\theta-\phi)\,d\theta\,d\phi=\frac{2a^2}{\pi^2}\int_0^{\pi}(\sin^2\theta-\theta\sin\theta\cos\theta)\,d\theta$$

$$=\frac{3a^2}{2\pi}.$$

## MISCELLANEOUS.

172. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If $\phi$ and $\psi$ are small angles, show that an approximate value of $\phi/\psi$ is

$$\tfrac{2}{3}\frac{\sin\phi}{\sin\psi}+\tfrac{1}{3}\frac{\tan\phi}{\tan\psi}-\tfrac{1}{180}(\phi^2-\psi^2)(9\phi^2-\psi^2).$$

Solution by G. W. GREENWOOD, Dunbar, Pa.

$$A=\frac{\sin\phi}{\sin\psi}=\frac{\phi}{\psi}\left(1-\frac{\phi^2}{6}+\frac{\psi^2}{6}+\frac{\phi^4}{120}-\frac{\phi^2\,\psi^2}{36}+\frac{7\psi^4}{360}\cdots\right)$$

$$B=\frac{\cos\phi}{\cos\psi}=1+\frac{\phi^2}{2}-\frac{\psi^2}{2}+\frac{5\phi^4}{24}-\frac{\phi^2\,\psi^2}{4}+\frac{\psi^4}{24}\cdots$$

$$\frac{\tan\phi}{\tan\psi}=AB=\frac{\phi}{\psi}\left(1+\frac{\phi^2}{3}-\frac{\psi^2}{3}+\frac{2\phi^4}{15}-\frac{\phi^2\,\psi^2}{9}-\frac{\psi^4}{45}\cdots\right)$$

$$\therefore\ \frac{2}{3}\frac{\sin\phi}{\sin\psi}+\frac{1}{3}\frac{\tan\phi}{\tan\psi}=\frac{\phi}{\psi}\left(1+\frac{\phi^4}{20}-\frac{\phi^2\,\psi^2}{18}+\frac{\psi^4}{180}\cdots\right)$$

$$=\frac{\phi}{\psi}[1+\tfrac{1}{180}(\phi^2-\psi^2)(9\phi^2-\psi^2)].$$

The problem does not appear correct.

Solved in a similar way and with the same result by G. B. M. Zerr.

## PROBLEMS FOR SOLUTION.

### ALGEBRA.

301. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$A$ is at Philadelphia, $B$ at Chicago. $A$'s personal equation is $e$; $B$'s is $E$. When a star crosses $A$'s meridian at time $t_1$=8 hours, 33 minutes, 24 seconds, he presses a button, telegraphing the fact to $B$, who receives it at time $t_2$=7 hours, 43 minutes, 23 seconds. When it crosses $B$'s meridian at time $T_2$=8 hours, 33 minutes, 10 seconds, he telegraphs $A$, who receives it at time $T_1$=9 hours, 23 minutes, 11 seconds. They now exchange places, and on the second day following, $B$ observes the transit at time $t'_1$=8 hours, 33 minutes, 26 seconds, and $A$ gets the information at Chicago at time $t'_2$= 7 hours, 43 minutes, 25 seconds. It crosses $A$'s meridian at time $T'_2$=8 hours, 33 minutes, 12 seconds, and $B$ gets the information at time $T'_1$=9 hours, 23 minutes, 13 seconds. Find the difference of longitude between Philadelphia and Chicago.

302. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that the system of equations

$$xu-5yv=2,$$
$$xv+yu=1,$$

has no integral solution in $x$, $y$, $u$, $v$ except those for which one of the unknowns is zero.

## GEOMETRY.

335. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Determine analytically, the point where three lines in a plane appear of equal length.

## CALCULUS.

260. Proposed by V. M. SPUNAR, East Pittsburg, Pa.

A natural equation of a surface may be defined as an equation in which the differential quotients of the principal radius, $\rho$, of curvature to the element of arc in the direction of the principal curvature are shown as a function of $\rho$, $\frac{d^n\rho}{ds}=F(\rho)$. Required the natural equation of the whole surface of second power.

261. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\sum_{x=1}^{x=\infty}\frac{1}{a+2bx^2+cx^4}=\frac{\pi}{\sqrt{[8ac(\sqrt{ac}+b)]}}-\frac{1}{2a}$, where $ac>b^2$.

262. Proposed by H. SCHAFFER, Fayetteville, Ark.

Prove that the circle is the only plane curve of constant curvature.

## MECHANICS.

217. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Given, the mean distance from earth to sun, $1.49\times10^{15}$ centimeters; radius of the earth, $6.37\times10^{8}$ centimeters; velocity of the earth in its orbit, $2.96\times10^{6}$ centimeters per second; velocity of rotation of a point on the equator, $4.63\times10^{4}$ centimeters per second; mass of the earth, $6.14\times10^{27}$ grams; find (1) the total energy of the earth in ergs; (2) the angular velocity of the earth on its axis; and (3) the angular velocity of the earth around the sun.

218. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Cut a uniform, circular cylinder by two planes whose line of intersection is without the cylinder. The centroid $G$ of the surface of the portion of the cylinder thus cut off lies in a plane elliptic section, in which plane also lies the line of intersection aforesaid. $C$ is the center of the ellipse, and the pole of the intersection line with reference to this ellipse is $X$. Show (1) that $C$, $X$, and $G$ are collinear, and (2) that $XC=2CG$.

## NUMBER THEORY AND DIOPHANTINE ANALYSIS.

153. Proposed by LLOYD HOLSINGER, A. B., 227 Fredonia Avenue, Peoria, Ill.

If we represent by $(k, l)$ the greatest common divisor of $k$ and $l$, and by $\phi(k)$ the number of integers prime to $k$ and not greater than $k$, we have

$$\begin{vmatrix} (1,1) & (1,2) & (1,3) & \dots & (1,n) \\ (2,1) & (2,2) & (2,3) & \dots & (2,n) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (n,1) & (n,2) & (n,3) & \dots & (n,n) \end{vmatrix} = \phi(1).\phi(2).\phi(3)\dots\phi(n).$$

## AVERAGE AND PROBABILITY.

198. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Find the average length of a hole at random through a given cylinder.

## MISCELLANEOUS.

178. Proposed by V. M. SPUNAR, Mechanical and Civil Engineer, East Pittsburg, Pa.

Find the sum of the series, $\frac{\sin x}{m^2+1} - \frac{2\sin 2x}{m^2+4} + \frac{3\sin 3x}{m^2+9} - \dots$ to infinity.

## ERRATA.

Page 100, line 2 from bottom, for "$(t_m, s_m)$" read $\rho(t_m, s_m)$.

Page 101, line 1 from top, for "positive" read negative; for "negative" read positive.

Page 101, line 12 from top, for "$(t, \rho(t, P(t, r)))$" read $\rho(t, P(t, r))$.

Page 101, line 13 from bottom, for "$I$" read $\delta$ in each case.

Page 102, line 4 from top, for "modulo 1" read modulo $l$.

Page 102, line 14 from top, for "$\rho(t, s) > r_2$" read $\rho(t, s) \geq r_2$.

Page 102, lines 8, 9, and 11 from bottom, the number (3) should go with the equation of line 11, and (4) should go with the equations of lines 8 and 9, which should be braced.

Page 102, line 8 from bottom, for "$y(t+h) - \lambda x'(t+h)$" read $y(t+h) + \lambda x'(t+h)$.

Page 103, line 10 from bottom, for "$x(t+\overline{h}) - y'(t+\overline{h})$" read $x(t+\overline{h}) - \overline{\lambda} y'(t+\overline{h})$.

Page 103, line 3 from bottom, for "$\pi$" read $\rho$.

## NOTE.

The radicals on pp. 76-77 are connected by the relation

$$8\sqrt{C_\pm} = \sqrt{D_+}.(a - \sqrt{B}) \pm \sqrt{D_-}.(a + \sqrt{B}).$$

This relation invalidates a criticism of the accuracy of the results found.

L. E. D.

8—9

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XV. AUGUST-SEPTEMBER, 1908. NOS. 8-9.

# ON MATHEMATICAL INDUCTION.

By DR. J. W. A. YOUNG, The University of Chicago.

## I. Its Function and Place.

In the secondary school, the pupil in mathematics is becoming familiar with this fundamental type of thought by working in it and applying it. He thinks mathematics, but not *about* mathematics. In college, when the student begins to philosophize, he may, in addition to working in and with mathematics, also begin to think about it, to analyze and classify its processes of thought, to seek its essential characteristics and the lines of demarcation between it and other subjects.

He will learn that many mathematicians see the distinctive marks of their science, not in its subject matter, not in numbers, points, lines and symbols, but in the mode of thinking which is used.* He will find the definition, "Mathematics is the science which draws necessary conclusions,"† a good expression of this conception of the subject. It identifies mathematics with deductive reasoning, and accounts for the peculiar certainty and accuracy which, in his experience, has been its distinctive characteristic.

But when he thinks of the way in which he works out mathematics, solves problems, does "original" work of any description, he sees that induction plays an important part in this aspect of mathematics.

Further, as his acquaintance with the subject matter of mathematics widens, he will find numerous instances in which concepts are extended and theorems generalized. He will frequently find that the course of development is from the particular to the general, and in such cases he will often find employed a method of reasoning called *mathematical induction*,‡ which shares with non-mathematical induction the peculiarity of generalizing from particular instances, but which nevertheless, like other mathematics, produces that unhesitating confidence in the absolute accuracy of the result

*For an instructive discussion of various definitions of mathematics, see Bocher, *Bulletin of American Mathematical Society*, 1904, p. 115.

†Peirce, *American Journal of Mathematics*, Vol. IV.

‡Also, *Complete Induction*, and, in German, *der Kaestnerische Schluss*, although published by Pascal in 1662 (Cantor, *Gesch. d. Math.*, II. p. 684) long before the time of Kaestner (1719—1800).